

# On Stanley's Partition Function

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**Abstract.** Stanley defined a partition function  $t(n)$  as the number of partitions  $\lambda$  of  $n$  such that the number of odd parts of  $\lambda$  is congruent to the number of odd parts of the conjugate partition  $\lambda'$  modulo 4. We show that  $t(n)$  equals the number of partitions of  $n$  with an even number of hooks of even length. We derive a closed-form formula for the generating function for the numbers  $p(n) - t(n)$ . As a consequence, we see that  $t(n)$  has the same parity as the ordinary partition function  $p(n)$  for any  $n$ . A simple combinatorial explanation of this fact is also provided.

**Keywords:** partition function, Jacobi's triple product identity, hook length.

**AMS Mathematical Subject Classifications:** 05A17.

## 1 Introduction

This note is concerned with the partition function  $t(n)$  introduced by Stanley [7, 8]. We shall give a combinatorial interpretation of  $t(n)$  in terms of hook lengths and shall prove that  $t(n)$  and the partition function  $p(n)$  have the same parity. Moreover, we compute the generating function for  $p(n) - t(n)$  and related generating functions.

We shall adopt the common notation on partitions in Andrews [1] or Andrews and Eriksson [3]. A partition  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_r)$  of a nonnegative integer  $n$  is a nonincreasing sequence of nonnegative integers such that the sum of the components  $\lambda_i$  equals  $n$ . A part is meant to be a positive component, and the number of parts of  $\lambda$  is called the length, denoted  $l(\lambda)$ . The conjugate partition of  $\lambda$  is defined by  $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_t)$ , where  $\lambda'_i$  ( $1 \leq i \leq t$ ,  $t = l(\lambda)$ ) is the number of parts in  $(\lambda_1, \lambda_2, \dots, \lambda_r)$  which are greater than or equal to  $i$ . The number of odd parts in  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$  is denoted by  $\mathcal{O}(\lambda)$ .

For  $|q| < 1$ , the  $q$ -shifted factorial is defined by

$$(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n \geq 1,$$

and

$$(a; q)_\infty := (1 - a)(1 - aq)(1 - aq^2) \cdots.$$

Stanley [7, 8] introduced the partition function  $t(n)$  as the number of partitions  $\lambda$  of  $n$  such that  $\mathcal{O}(\lambda) \equiv \mathcal{O}(\lambda') \pmod{4}$ , and obtained the following formula

$$t(n) = \frac{1}{2} (p(n) + f(n)), \quad (1.1)$$

where  $p(n)$  is the number of partitions of  $n$  and

$$\sum_{n=0}^{\infty} f(n)q^n = \prod_{i \geq 1} \frac{(1 + q^{2i-1})}{(1 - q^{4i})(1 + q^{4i-2})^2}. \quad (1.2)$$

Andrews [2] obtained the following closed-form formula for the generating function of  $t(n)$

$$\sum_{n=0}^{\infty} t(n)q^n = \frac{(q^2; q^2)_{\infty}^2 (q^{16}; q^{16})_{\infty}^5}{(q; q)_{\infty} (q^4; q^4)_{\infty}^5 (q^{32}; q^{32})_{\infty}^2}. \quad (1.3)$$

He also derived the congruence relation

$$t(5n + 4) \equiv 0 \pmod{5}. \quad (1.4)$$

In this note, we shall consider the complementary partition function of  $t(n)$ , namely, the partition function  $u(n) = p(n) - t(n)$ , which is the number of partitions  $\lambda$  of  $n$  such that  $\mathcal{O}(\lambda) \not\equiv \mathcal{O}(\lambda') \pmod{4}$ . We obtain a closed-form formula for the generating function of  $u(n)$  which implies that Stanley's partition function  $t(n)$  and ordinary partition function  $p(n)$  have the same parity for any  $n$ . We also present a simple combinatorial explanation of this fact. Then we derive formulas for the generating functions for the numbers  $u(4n)$ ,  $u(4n+1)$ ,  $u(4n+2)$  and  $u(4n+3)$  which are analogous to the formulas for the partition function  $t(n)$  due to Andrews [2]. In the last section, we find combinatorial interpretations for  $t(n)$  and  $u(n)$  in terms of hooks of even length.

## 2 The generating function formula

We shall derive a formula for the partition function  $u(n) = p(n) - t(n)$ . The proof is similar to Andrews' proof of (1.3) for  $t(n)$ . As a consequence, one sees that  $t(n)$  and  $p(n)$  have the same parity for any  $n$ . This fact also has a simple combinatorial interpretation. We shall also compute the generating functions for the numbers  $u(4n)$ ,  $u(4n+1)$ ,  $u(4n+2)$  and  $u(4n+3)$ .

**Theorem 2.1** *We have*

$$\sum_{n=0}^{\infty} u(n)q^n = \frac{2q^2(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2 (q^{32}; q^{32})_{\infty}^2}{(q; q)_{\infty} (q^4; q^4)_{\infty}^5 (q^{16}; q^{16})_{\infty}} \quad (2.5)$$

*Proof.* By (2.18) and (1.2), we find

$$\begin{aligned}
\sum_{n=0}^{\infty} u(n)q^n &= \frac{1}{2} \left( \frac{1}{(q; q)_{\infty}} - \frac{(-q; q^2)_{\infty}}{(q^4; q^4)_{\infty}(-q^2; q^4)_{\infty}^2} \right) \\
&= \frac{1}{2} \left( \frac{(-q; q^2)_{\infty}}{(q^4; q^4)_{\infty}(q^2; q^4)_{\infty}^2} - \frac{(-q; q^2)_{\infty}}{(q^4; q^4)_{\infty}(-q^2; q^4)_{\infty}^2} \right) \\
&= \frac{(-q; q^2)_{\infty}}{2(q^4; q^4)_{\infty}^2 (q^2; q^4)_{\infty}^2 (-q^2; q^4)_{\infty}^2} ((q^4; q^4)_{\infty}(-q^2; q^4)_{\infty}^2 - (q^4; q^4)_{\infty}(q^2; q^4)_{\infty}^2).
\end{aligned}$$

Using Jacobi's triple product identity [4, p.10]

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq; q^2)_{\infty} (-q/z; q^2)_{\infty} (q^2; q^2)_{\infty}, \quad (2.6)$$

we see that

$$(q^4; q^4)_{\infty} (-q^2; q^4)_{\infty}^2 = \sum_{n=-\infty}^{\infty} q^{2n^2} \quad (2.7)$$

and

$$(q^4; q^4)_{\infty} (q^2; q^4)_{\infty}^2 = \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}. \quad (2.8)$$

Clearly,

$$\sum_{n=-\infty}^{\infty} q^{2n^2} - \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} = 2 \sum_{n=-\infty}^{\infty} q^{2(2n+1)^2} \quad (2.9)$$

It follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} u(q)q^n &= \frac{(-q; q^2)_{\infty}}{(q^4; q^4)_{\infty}^2 (q^2; q^4)_{\infty}^2 (-q^2; q^4)_{\infty}^2} \sum_{n=-\infty}^{\infty} q^{2(2n+1)^2} \\
&= \frac{q^2(-q; q^2)_{\infty}}{(q^4; q^4)_{\infty}^2 (q^4; q^8)_{\infty}^2} \sum_{n=-\infty}^{\infty} q^{8n^2+8n}.
\end{aligned} \quad (2.10)$$

Jacobi's triple product identity yields

$$\sum_{n=-\infty}^{\infty} q^{8n^2+8n} = (-q^{16}; q^{16})_{\infty} (-1; q^{16})_{\infty} (q^{16}; q^{16})_{\infty}. \quad (2.11)$$

Observe that

$$(-1; q^{16})_{\infty} = 2(-q^{16}; q^{16})_{\infty}. \quad (2.12)$$

In view of (2.10), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} u(q)q^n &= \frac{2q^2(-q^{16}; q^{16})_{\infty} (-q^{16}; q^{16})_{\infty} (-q; q^2)_{\infty} (q^{16}; q^{16})_{\infty}}{(q^4; q^4)_{\infty}^2 (q^4; q^8)_{\infty}^2} \\
&= \frac{2q^2(q^{32}; q^{32})_{\infty} (-q; q^2)_{\infty} (-q^{16}; q^{16})_{\infty}}{(q^4; q^4)_{\infty}^2 (q^4; q^8)_{\infty}^2}.
\end{aligned}$$

Now,

$$(-q; q^2)_\infty = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty (q^4; q^4)_\infty}, \quad (2.13)$$

$$(q^4; q^8)_\infty = \frac{(q^4; q^4)_\infty}{(q^8; q^8)_\infty} \quad (2.14)$$

and

$$(-q^{16}; q^{16})_\infty = \frac{(q^{32}; q^{32})_\infty}{(q^{16}; q^{16})_\infty}. \quad (2.15)$$

Consequently,

$$\begin{aligned} \sum_{n=0}^{\infty} u(q) q^n &= \frac{2q^2 (q^{32}; q^{32})_\infty (q^8; q^8)_\infty^2 (q^2; q^2)_\infty^2 (q^{32}; q^{32})_\infty}{(q^4; q^4)_\infty^2 (q^4; q^4)_\infty^2 (q; q)_\infty (q^4; q^4)_\infty (q^{16}; q^{16})_\infty} \\ &= \frac{2q^2 (q^2; q^2)_\infty^2 (q^8; q^8)_\infty^2 (q^{32}; q^{32})_\infty^2}{(q; q)_\infty (q^4; q^4)_\infty^5 (q^{16}; q^{16})_\infty}. \end{aligned}$$

This completes the proof. ■

**Corollary 2.2** *For  $n \geq 0$ ,*

$$t(n) \equiv p(n) \pmod{2}.$$

We remark that there is a simple combinatorial explanation of the above parity property. First, we observe that for any partition  $\lambda$  of  $n$ ,

$$O(\lambda) \equiv O(\lambda') \pmod{2} \quad (2.16)$$

because we have both  $O(\lambda) \equiv n \pmod{2}$  and  $O(\lambda') \equiv n \pmod{2}$ . By the definition of  $u(n)$  and the relation (2.16), we see that  $u(n)$  equals the number of partitions of  $n$  such that

$$O(\lambda) - O(\lambda') \equiv 2 \pmod{4}. \quad (2.17)$$

Suppose  $\lambda$  is a partition counted by  $u(n)$ . From (2.17) it is evident that its conjugation  $\lambda'$  is also counted by  $u(n)$ . Once more, from (2.17) we deduce that  $O(\lambda)$  and  $O(\lambda')$  are not equal, so that  $\lambda$  is different from  $\lambda'$ . Thus we reach the conclusion that  $u(n)$  must be even, and so  $t(n)$  has the same parity as  $p(n)$  since  $p(n) = t(n) + u(n)$ .

From (1.1) it follows that

$$u(n) = p(n) - t(n) = \frac{p(n) - f(n)}{2}. \quad (2.18)$$

So we have the following congruence relation.

**Corollary 2.3** *For  $n \geq 0$ ,*

$$f(n) \equiv p(n) \pmod{4}.$$

Theorem 2.1 enables us to derive the generating functions for  $u(4n + i)$ , where  $i = 0, 1, 2, 3$ . Andrews [2] has obtained formulas for the generating functions of  $t(4n + i)$  for  $i = 0, 1, 2, 3$ .

**Theorem 2.4** *We have*

$$\begin{aligned}\sum_{n=0}^{\infty} u(4n)q^n &= 2q^2(q^{16}; q^{16})_{\infty}(-q; q^{16})_{\infty}(-q^{15}; q^{16})_{\infty}V(q) \\ \sum_{n=0}^{\infty} u(4n+1)q^n &= 2q(q^{16}; q^{16})_{\infty}(-q^3; q^{16})_{\infty}(-q^{13}; q^{16})_{\infty}V(q) \\ \sum_{n=0}^{\infty} u(4n+2)q^n &= 2(q^{16}; q^{16})_{\infty}(-q^7; q^{16})_{\infty}(-q^9; q^{16})_{\infty}V(q) \\ \sum_{n=0}^{\infty} u(4n+3)q^n &= 2(q^{16}; q^{16})_{\infty}(-q^5; q^{16})_{\infty}(-q^{11}; q^{16})_{\infty}V(q)\end{aligned}$$

where

$$V(q) = \frac{(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q; q)_{\infty}^5 (q^4; q^4)_{\infty}}.$$

*Proof.* By Theorem 2.1, we find

$$\begin{aligned}\sum_{n=0}^{\infty} u(n)q^n &= \frac{2q^2(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}}V(q^4) \\ &= \frac{2q^2(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}V(q^4)\end{aligned}$$

Since

$$\frac{1}{(q; q^2)_{\infty}} = (-q; q)_{\infty} \quad (2.19)$$

and

$$(q^2; q^2)_{\infty} = (q; q)_{\infty}(-q; q)_{\infty}, \quad (2.20)$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty} u(n)q^n &= 2q^2(q; q)_{\infty}(-q; q)_{\infty}(-q; q)_{\infty}V(q^4) \\ &= q^2(q; q)_{\infty}(-1; q)_{\infty}(-q; q)_{\infty}V(q^4).\end{aligned}$$

Using Jacobi's triple product identity, we get

$$(q; q)_{\infty}(-1; q)_{\infty}(-q; q)_{\infty} = \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}}. \quad (2.21)$$

Thus we have

$$\sum_{n=0}^{\infty} u(n)q^n = q^2 \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} V(q^4) = 2q^2 \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} V(q^4). \quad (2.22)$$

It is easy to check that

$$\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \sum_{n=-\infty}^{\infty} q^{2n^2-n}. \quad (2.23)$$

In view of (2.22), we get

$$\begin{aligned} \sum_{n=0}^{\infty} u(n)q^n &= 2q^2 \sum_{n=-\infty}^{\infty} q^{2n^2-n} V(q^4) \\ &= 2q^2 \sum_{i=0}^3 \sum_{k=-\infty}^{\infty} q^{2(4k+i)^2-(4k+i)} V(q^4). \end{aligned} \quad (2.24)$$

For  $i = 0$ , extracting the terms of the form  $q^{4j+2}$  in (2.24) for  $j = 0, 1, 2, \dots$ , we obtain

$$\sum_{n=0}^{\infty} u(4n+2)q^{4n+2} = 2q^2 \sum_{j=-\infty}^{\infty} q^{32j^2-4j} V(q^4).$$

Again, Jacobi's triple product identity gives

$$\sum_{j=-\infty}^{\infty} q^{32j^2-4j} = (q^{64}; q^{64})_{\infty} (-q^{28}; q^{64})_{\infty} (-q^{36}; q^{64})_{\infty}. \quad (2.25)$$

Hence we get

$$\sum_{n=0}^{\infty} u(4n+2)q^{4n+2} = 2q^2 (q^{64}; q^{64})_{\infty} (-q^{28}; q^{64})_{\infty} (-q^{36}; q^{64})_{\infty} V(q^4),$$

which simplifies to

$$\sum_{n=0}^{\infty} u(4n+2)q^n = 2(q^{16}; q^{16})_{\infty} (-q^7; q^{16})_{\infty} (-q^9; q^{16})_{\infty} V(q).$$

The remaining cases can be verified using similar arguments. This completes the proof. ■

### 3 Combinatorial interpretations for $t(n)$ and $u(n)$

In [7, Proposition 3.1], Stanley found three partition statistics that have the same parity as  $(\mathcal{O}(\lambda) - \mathcal{O}(\lambda'))/2$ , and gave several combinatorial interpretations for  $t(n)$ . We shall present combinatorial interpretations of partition functions  $t(n)$  and  $u(n)$  in terms of the number of hooks of even length. For the definition of hook lengths, see Stanley [6, p. 373]. A hook of even length is called an even hook. The following theorem shows that the number of even hooks has the same parity as  $(\mathcal{O}(\lambda) - \mathcal{O}(\lambda'))/2$ .

**Theorem 3.1** *For any partition  $\lambda$  of  $n$ ,  $\mathcal{O}(\lambda) \equiv \mathcal{O}(\lambda') \pmod{4}$  if and only if  $\lambda$  has an even number of even hooks.*

*Proof.* We use induction on  $n$ . It is clear that Theorem 3.1 holds for  $n = 1$ . Suppose that it is true for all partitions of  $n$ . We aim to show that the conclusion also holds for all partitions of  $n + 1$ . Let  $\lambda$  be a partition of  $n + 1$  and  $v = (i, j)$  be any an inner corner of the Young diagram of  $\lambda$ , that is, the removal of the square  $v$  gives a Young diagram of a partition of  $n$ . Let  $\lambda^-$  denote the partition obtained by removing the square  $v$  from the Young diagram of  $\lambda$ . We use  $H_e(\lambda)$  to denote the number of squares with even hooks in the Young diagram of  $\lambda$ . We claim that

$$H_e(\lambda) \equiv H_e(\lambda^-) \pmod{2} \iff \lambda_i \equiv \lambda'_j \pmod{2}. \quad (3.26)$$

Let  $\mathcal{T}(\lambda, v)$  denote the set of all squares in the Young diagram of  $\lambda$  which are in the same row as  $v$  or in the same column as  $v$ . After removing the square  $v$  from the Young diagram of  $\lambda$ , the hook lengths of the squares in  $\mathcal{T}(\lambda, v)$  have decreased by one. Meanwhile, the hook lengths of other squares remain the same. Furthermore, if  $\lambda_i$  and  $\lambda'_j$  have the same parity, then the number of squares in  $\mathcal{T}(\lambda, v)$  is even. This implies that the parity of the number of squares in  $\mathcal{T}(\lambda, v)$  of even hook lengths coincides with the parity of the number of squares in  $\mathcal{T}(\lambda, v)$  with odd hook lengths. Similarly, for the case when  $\lambda_i$  and  $\lambda'_j$  have different parities, it can be shown that the number of squares in  $\mathcal{T}(\lambda, v)$  of even hook length is of opposite parity to the number of squares in  $\mathcal{T}(\lambda, v)$  of odd hook length. Hence we arrive at (3.26).

By the inductive hypothesis, we see that  $\mathcal{O}(\lambda^-) \equiv \mathcal{O}((\lambda^-)') \pmod{4}$  if and only if  $H_e(\lambda^-)$  is even. For any inner corner  $v = (i, j)$  of  $\lambda$ , if  $\lambda_i \equiv \lambda'_j \pmod{2}$ , then  $\mathcal{O}(\lambda) \equiv \mathcal{O}(\lambda') \pmod{4}$  if and only if  $\mathcal{O}(\lambda^-) \equiv \mathcal{O}((\lambda^-)') \pmod{4}$ . By (3.26), we find that in this case,  $H_e(\lambda)$  and  $H_e(\lambda^-)$  have the same parity. Thus the assertion holds for any partition  $\lambda$  of  $n + 1$ . The case that  $\lambda_i \not\equiv \lambda'_j \pmod{2}$  can be justified in the same manner. This completes the proof.  $\blacksquare$

From Theorem 3.1, we obtain a combinatorial interpretation for Stanley's partition function  $t(n)$ , which can be recast as a combinatorial interpretation for  $u(n)$ .

**Theorem 3.2** *The partition function  $t(n)$  is equal to the number of partitions of  $n$  with an even number of even hooks, and the partition function  $u(n)$  is equal to the number of partitions of  $n$  with an odd number of even hooks.*

Combining Theorem 2.1 and Theorem 3.2 we have the following consequence.

**Corollary 3.3** *For any  $n$ , the number of partitions of  $n$  with an odd number of even hooks is always even.*

Since  $f(n) = t(n) - u(n)$ , we see that  $f(n)$  can be interpreted as the signed counting of partitions of  $n$  with respect to the number of even hooks, as formally stated below.

**Theorem 3.4** *The function  $f(n)$  equals the number of partitions of  $n$  with an even number of even hooks minus the number of partitions of  $n$  with an odd number of even hooks.*

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